# Three-dimensional Analytical Force-free MHD Equilibria

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Three-dimensional analytical force-free magnetohydrostatic equilibria may be obtained by separation of variables with respect to cylindrical coordinates. These equilibria are then used to study numerically flux surfaces of various three-dimensional configurations.

## I. Representation of Force-free Fields

Let $r, \varphi, z$ be cylindrical coordinates and suppose separation of variables ** with respect to $\varphi$ and $z$ for the components $\bar{B}_r$, $\bar{B}_\varphi$, $\bar{B}_z$ of the magnetic field $\boldsymbol{B}$:

$$\bar{B}_r = B_r \exp i(n\varphi + kz), \quad \text{etc.} \tag{1}$$

The equation for force-free equilibria,

$$\operatorname{curl} \boldsymbol{B} = \gamma \boldsymbol{B}, \tag{2}$$

where $\gamma$ is a constant, then reduces to

$$\begin{aligned} &\frac{in}{r} B_z - ik B_\varphi = \gamma B_r, \\ &ik B_r - B_z' = \gamma B_\varphi, \\ &\frac{1}{r}(r B_\varphi)' - \frac{in}{r} B_r = \gamma B_z, \end{aligned} \tag{3}$$

where $' = \mathrm{d}/\mathrm{d}r$, whose solutions reduce to Bessel functions as follows.

For $k^2 \neq \gamma^2$ we eliminate $B_r$ and $B_\varphi$ to find

$$\begin{aligned} B_\varphi &= \frac{1}{\gamma^2 - k^2}\left(-\frac{nkB_z}{r} - \gamma B_z'\right), \\ B_r &= \frac{i}{\gamma^2 - k^2}\left(\frac{\gamma n B_z}{r} + k B_z'\right), \\ B_z'' &+ \frac{B_z'}{r} + \left(\gamma^2 - k^2 - \frac{n^2}{r^2}\right) B_z = 0, \\ B_z &= Z_n(\sqrt{\gamma^2 - k^2}\, r). \end{aligned} \tag{4}$$

For $k = \gamma$

$$B_z = b_- r^{-n},$$

$$B_\varphi = b_{\varphi -} r^{-n+1} + b_-\left[-\frac{\gamma}{2(n-1)} r^{-n+1} + \frac{n}{2\gamma} r^{-n-1}\right], \quad n \neq 1,$$

$$B_\varphi = b_{\varphi -} + b_-\left[\gamma \log r + \frac{1}{2\gamma} r^{-2}\right], \quad n = 1,$$

$$B_r = -i\left(B_\varphi - \frac{n B_z}{\gamma r}\right),$$

and for $k = -\gamma$

$$B_z = b_+ r^n,$$

$$B_\varphi = b_{\varphi +} r^{n+1} + b_+\left[\frac{\gamma}{2(n+1)} r^{n+1} + \frac{n}{2\gamma} r^{n-1}\right],$$

$$B_r = i\left(B_\varphi + \frac{n B_z}{\gamma r}\right).$$

## II. Applications

### II.1. Axially Symmetric Equilibria

Here, we describe the relation to the usual equilibrium representation by a flux function $T$ in axial symmetry. A normalized form of this function (for details, see, for example, [3]) satisfies the equilibrium equation ($r = 1$ at the magnetic axis):

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) T = -\frac{4}{\iota_0^2}[f f' + r^2 p'].$$

For force-free equilibria with $\gamma = $ const we obtain

$$\gamma = \frac{\dot{J}}{\dot{\Phi}} = \iota_0\left(e + \frac{1}{e}\right) = -\frac{2}{\iota_0} f',$$

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** Separation of variables in spherical coordinates for axi-symmetric force-free equilibria was used in [1, 2].

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so that

$$f = 1 + \gamma_0 T, \quad \gamma_0 = -\frac{\iota_0^2}{2}\left(e + \frac{1}{e}\right),$$

$$f f' = \gamma_0 + \gamma_1 T, \quad \gamma_1 = \gamma_0^2,$$

and the relation between the flux function and the toroidal field component is

$$T = [B_\varphi r - 1]/\gamma_0 .$$

The separation ansatz for $T$ [3]

$$T = u(r) + v(r) \cos kz$$

is thus related to

$$B_\varphi = \overset{1}{Z} + \overset{2}{Z} \cos kz ,$$

where

$$\overset{1}{Z} = A J_0(\gamma r) + B Y_0(\gamma r),$$

$$\overset{2}{Z} = C J_0(\sqrt{\gamma^2 - k^2}\, r) + D Y_0(\sqrt{\gamma^2 - k^2}\, r),$$

by

$$(1 + \gamma_0 u)/r = \overset{1}{Z},$$

$$\gamma_0 v/r = \overset{2}{Z} .$$

These relations allow the coefficients $A, \ldots, D$ to be related to the equilibrium parameters of [3].

In Fig. 1 two examples are shown:

a) $\iota_0 = 1$, $e = 1$ $(\gamma = 2)$, $\gamma_0 = -1$, $k^2 = 21/8$. This choice of $k$ leads to approximately circular cross-section equilibria (in the notation of [4], [5], $S_{33} = S_{44} = 0$). The shear $(S = -2\pi^2 R^3 i/\iota)$ is weak, but its value on axis $S(0) = 13/32$ is sufficient to suppress the current-driven MHD $m = 1$ instability [6].

b) $\iota_0 = 1$, $e = 1$ $(\gamma = 2)$, $\gamma_0 = -1$, $k^2 = -3/8$. This choice of $k$ leads to a triangular deformation of the outer flux surfaces $(S_{33} = -1)$ in the form of a $D$-shape. The shear is larger than in case a), $S(0) = 61/32$, but still rather small.

### *II.2. Three-dimensional Equilibria*

A) Perturbation of axisymmetric equilibria with and without separatrix formation

Perturbing the equilibrium II.1 a) with a field $\tilde{\boldsymbol{B}}$ given by

$$\tilde{B}_z = \varepsilon J_n(2r) \cos n\varphi ,$$

one obtains the results shown in Fig. 2 for $n = 1$ and $n = 2$. A simple argument relating the closure of the field line which is to be the new magnetic axis to the resonant part of the perturbing field or of its interaction with the axisymmetric field shows that the excursion $\Delta r/R$ of the new magnetic axis is given by $(1 \leqq n \leqq 3)$

$$\Delta r/R \sim \varepsilon^{1/(4-n)} .$$

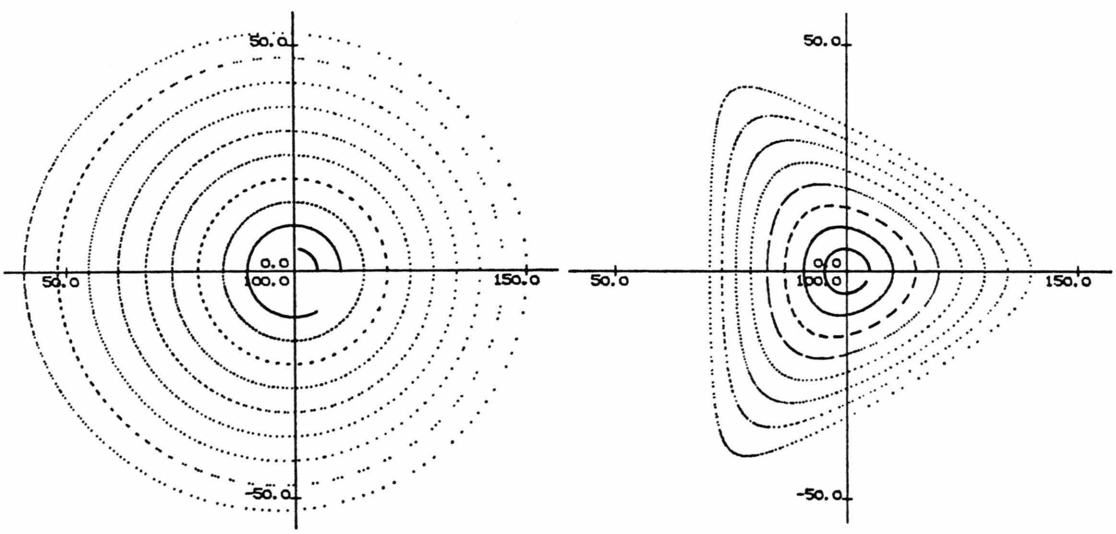


Fig. 1. Axially symmetric equilibria, a) $k^2 = +21/8$, $\iota_0 = 1.0$; b) $k^2 = -3/8$, $\iota_0 = 1.0$. Shown is a cross-section in the *r-z*-plane (*r* is normalized to 100 at the magnetic axis). A couple of field lines is followed a few hundred times around the torus to trace magnetic surfaces.

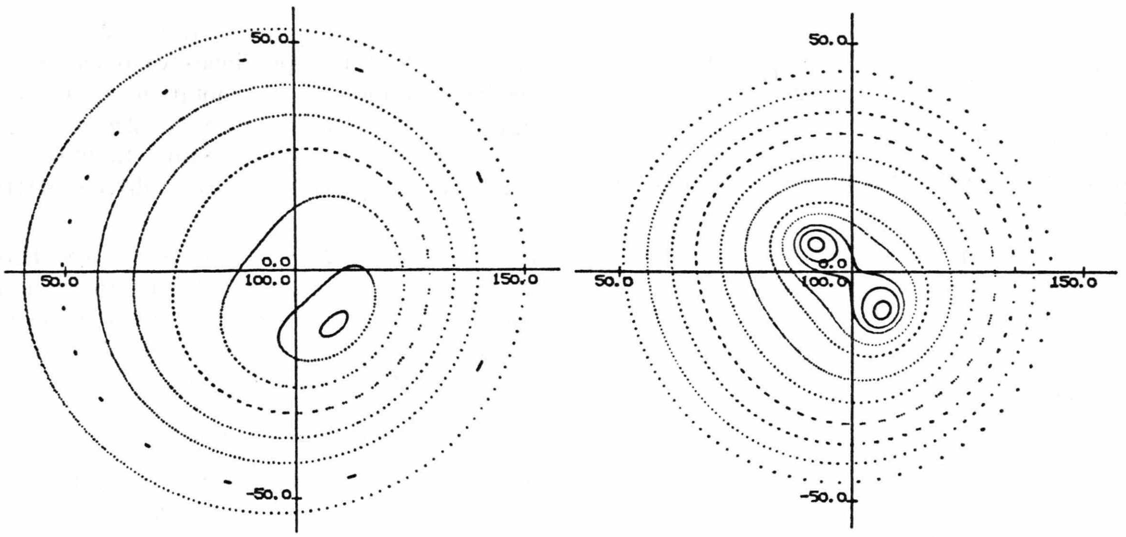


Fig. 2. Perturbation of axisymmetric equilibria with and without separatrix formation, a) $k^2 = +21/8$, $\iota_0 = 1.0$, $n = 1$, $\varepsilon = -0.01$; b) $k^2 = +21/8$, $\iota_0 = 1.0$, $n = 2$, $\varepsilon = -0.05$; Shown is a cross-section in the $r$-$z$-plane, which corresponds to $\varphi = 2\pi/8$.

For $n = 1$ and 2 we verified this relation in the range $0 \leqq \varepsilon \leqq 10^{-2}$ and $0 \leqq \varepsilon \leqq 5 \times 10^{-2}$, respectively.

For $n = 1$ no separatrix is formed and no isolation [7] is observed, so that the helical structure is smoothly embedded in the surrounding nearly axisymmetric equilibrium. This particular case was used as a test equilibrium for a 3D numerical MHD code [8].

For $n = 2$ a separatrix is formed but no isolation is apparent.

## B) Isolation

In the outer part of equilibrium II.1b) the rotational transform takes on the rational values 7/8 and 5/6. Perturbing the equilibrium with $n = 7$ and $n = 5$ fields $\tilde{\boldsymbol{B}}$ given by

$$\tilde{B}_z = \varepsilon_1 J_7(2r) \cos 7\varphi + \varepsilon_2 J_5(2r) \cos 5\varphi,$$
$$\varepsilon_1 = -50.0, \quad \varepsilon_2 = -0.80,$$

one obtains two distinct chains of islands at the above rational values (with 8 and 6 islands, respectively), as is shown in Figure 3. The structure of magnetic "surfaces" in the neighbourhood of these islands is as follows. While close to the inner chain the surfaces are smooth on its inside, they are "rough" on its outside. Further to the outside, field lines not confined by the outer chain of islands fill out a band around this chain and eventually get lost to the outside.

Varying $\varepsilon_1$ and $\varepsilon_2$, we were unable to generate a region of stochastic field lines embedded in smooth outer surfaces. An obvious conjecture in this context would be that the solution to a boundary value problem with smooth outer boundary of the domain considered would show no isolation.

## C) Model of the force-free $l = 2$ stellarator

Starting with an axially symmetric field $\boldsymbol{B}_{\text{ax}}$, we add a three-dimensional field $\bar{\boldsymbol{B}}$ which satisfies the following conditions:

a) There is a stagnation point of the transverse field at $r = 1$, $z = 0$, and no modulation of the main field is introduced at this point:

$$\bar{B}_z(1) = \bar{B}_r(1) = \bar{B}_\varphi(1) = 0. \tag{5}$$

Because of Eq. (1) a field with these properties is irrotational in the neighbourhood of the magnetic axis.

b) Near the magnetic axis the $l = 2$ stellarator field is given by (see, for example [9]) ($x = r - 1$)

$$B_r = -a(z \cos n\varphi - x \sin n\varphi),$$
$$B_z = -a(x \cos n\varphi + z \sin n\varphi), \tag{6}$$

which obviously yields another set of three conditions.

Expansion of a single $\bar{\boldsymbol{B}}$-field with given $k$ yields

$$\bar{B}_z = B_z'(1)\, x \cos n\varphi - k z B_z(1) \sin n\varphi,$$
$$\bar{B}_r = B_r'(1)\, x \sin n\varphi + k z B_r(1) \cos n\varphi. \tag{7}$$

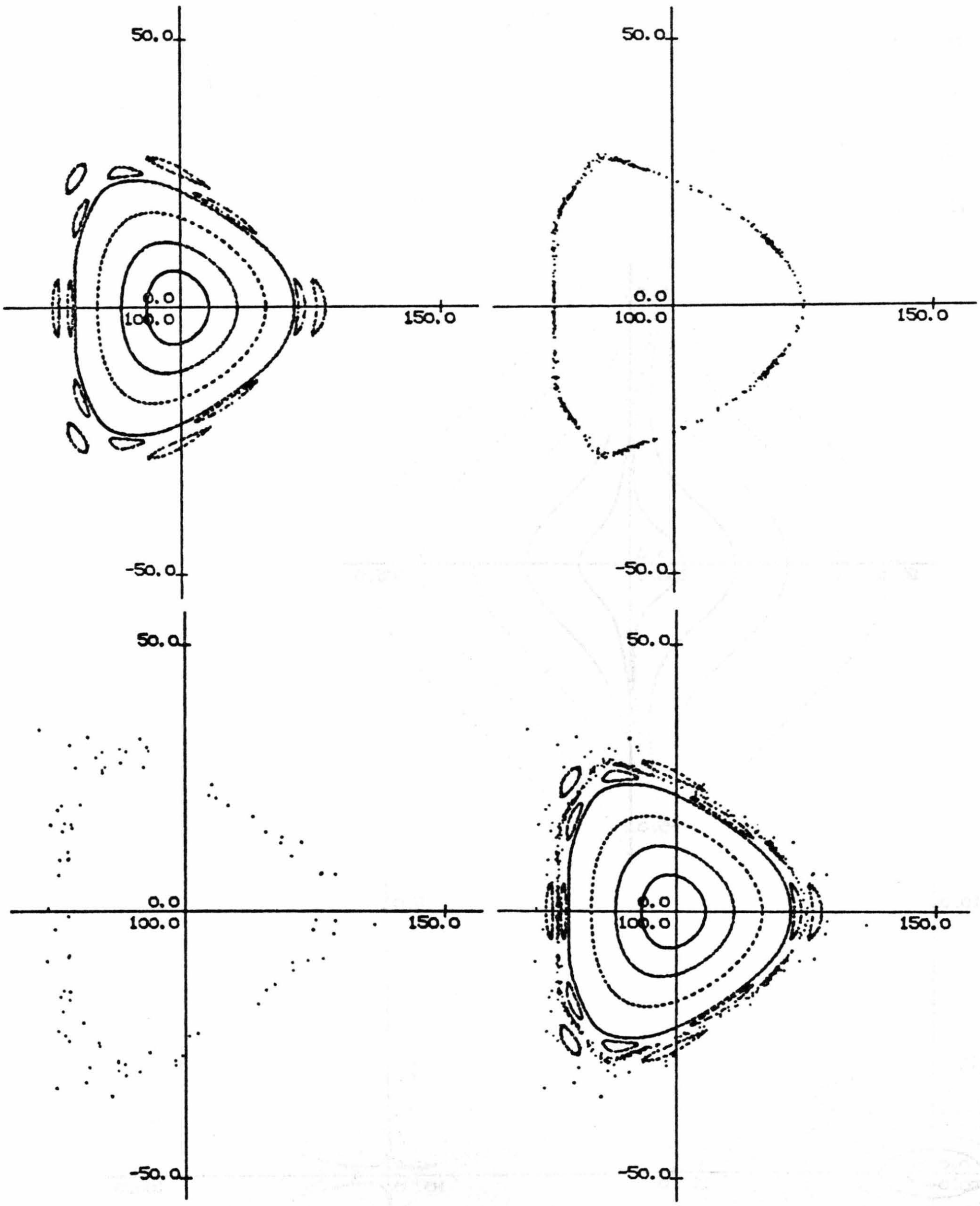


Fig. 3. Isolation with $k^2 = -3/8$, $n_1 = 7$, $n_2 = 5$, $\varepsilon_1 = -50.0$, $\varepsilon_2 = -0.80$. a) Inner closed surfaces and two sets of islands; b) "rough" surface between islands; c) outer field line which is not confined; d) combination of $a + b + c$. Shown is a cross-section in the $r$-$z$-plane at $\varphi = 0$.

Because of Eqs. (5), (6) three constituents of $\bar{\boldsymbol{B}}$ with different $k$ will thus be needed. Employing an ansatz

$$\bar{B}_z = \overset{1}{B}_z \cos(n\varphi + k_1 z) + \overset{0}{B}_z \cos n\varphi + \overset{-1}{B}_z \cos(n\varphi - k_1 z),$$

we find from Eqs. (5) and (6) the following conditions at $r = 1$:

$$\overset{1}{B}_z + \overset{0}{B}_z + \overset{-1}{B}_z = 0,$$

$$\overset{1}{B}_r + \overset{0}{B}_r + \overset{-1}{B}_r = 0,$$

$$\overset{1}{B}_\varphi + \overset{0}{B}_\varphi + \overset{-1}{B}_\varphi = 0\,,$$
$$\overset{1}{B}_z{}' + \overset{-1}{B}_z{}' + \overset{0}{B}_z{}' + k_1(\overset{1}{B}_z - \overset{-1}{B}_z) = 0\,,$$
$$\overset{1}{B}_r{}' + \overset{-1}{B}_r{}' + \overset{0}{B}_r{}' + k_1(\overset{1}{B}_r - \overset{-1}{B}_r) = 0\,,$$
$$\overset{1}{B}_r + \overset{1}{B}_z - \overset{-1}{B}_r - \overset{-1}{B}_z = 0\,,$$

which may be solved by

$$\overset{1}{B}_z{}' = \frac{1}{k_1}(\gamma^2 - k_1^2 - \gamma n)\,\overset{1}{B}_z\,,$$
$$\overset{-1}{B}_z = -\overset{1}{B}_z\,,$$
$$\overset{-1}{B}_z{}' = \overset{1}{B}_z{}'\,,$$

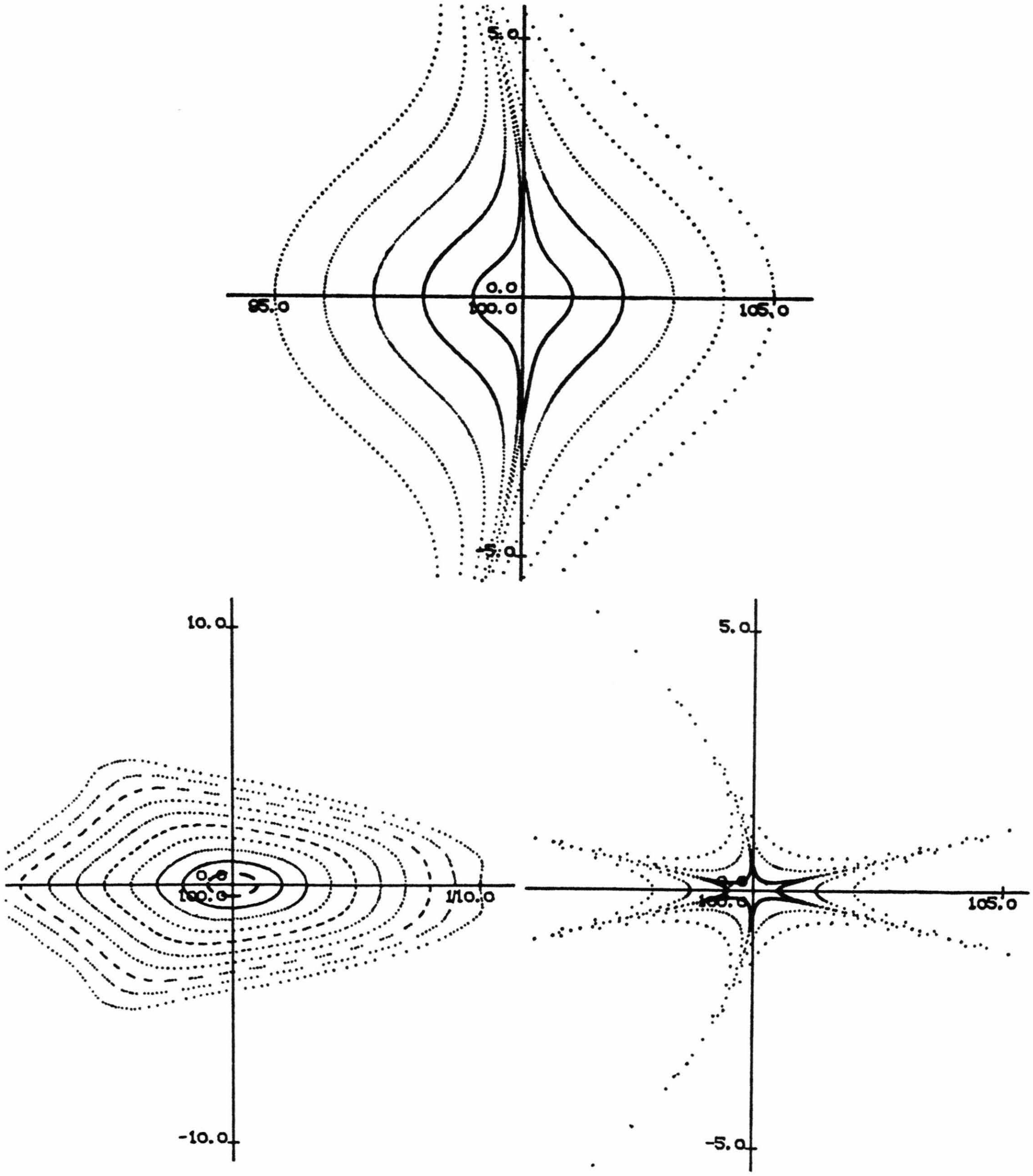


Fig. 4. Special stellarator equilibria: a) $k^2 = +3/8$, $n = 5$, $\varepsilon = 0.0043$, $\iota_0 = 0.0$; b) $k^2 = +3/8$, $n = 5$, $\varepsilon = 0.00168$, $\iota_0 = 1.0$; c) $k^2 = -3/16$, $n = 1$, $\varepsilon = 0.2225$, $\iota_0 = 1/3$. Shown is a cross-section in the $r$-$z$-plane at $\varphi = 0$.

$$\overset{0}{B}_z = 0\,,$$

$$\overset{0}{B}_z{}' = -\overset{1}{B}_z{}' - \overset{-1}{B}_z{}' - k_1(\overset{1}{B}_z - \overset{-1}{B}_z)\,.$$

With $\overset{1}{B}_z = 1$, the complete field configuration is then given by

$$\boldsymbol{B} = \boldsymbol{B}_{ax} + \varepsilon \overline{\boldsymbol{B}}\,,$$

where $\varepsilon$ governs the magnitude of the $l = 2$ component.

The analytical relation between the longitudinal current density $\gamma$, the rotational transform $\iota_0$ and the ellipticity $e$ (ratio of half-axes) near the magnetic axis is given by (see, for example [9])

$$\iota_0 = \frac{\gamma}{e + 1/e} \mp \frac{n}{2} \frac{(e-1)^2}{e^2 + 1}\,.$$

Examples for $n = 5$, $\gamma = 1$ are shown in Figure 4. For

$$e = \tfrac{1}{5}(6 \pm \sqrt{11})\,.$$

$\iota_0$ becomes zero for the case in which the $l = 2$ transform is subtractive. Since the integral

$$\oint \mathrm{d}l/B$$

is not stationary on the magnetic axis in this configuration (see, for example [10]), a change of the topology of flux surfaces occurs at this value of $e$, or equivalently, $\iota_0 = 0$, as is shown in Figure 4. On the other hand, in accordance with the theoretical result, no resonance is observed at $\iota_0 = 1$ (see Figure 4).

To demonstrate a higher order change of topology of the flux surfaces, we finally consider the example $n = 1$, $\gamma = \frac{1}{2}$ and $\iota_0 = \frac{1}{3}$. At this value of the rotational transform the closed flux surfaces break up into the pattern shown in Figure 4.


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